Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 53-2 (2020) 6730–6735

Numerical methods for construction of value functions in optimal control problems with infinite horizon

Alexander L. Bagno * **Alexander M. Tarasyev** **

* *Krasovskii Institute of Mathematics and Mechanics of UrB RAS, Ekaterinburg, Russia (e-mail: bagno.alexander@gmail.com).*

** *Krasovskii Institute of Mathematics and Mechanics of UrB RAS and Ural Federal University, Ekaterinburg, Russia (e-mail: tam@imm.uran.ru)*

Abstract: The article is devoted to the analysis of optimal control problems with infinite time horizon. These problems arise in economic growth models and in stabilization problems for dynamic systems. The problem peculiarity is a quality functional with an unbounded integrand which is discounted by an exponential index. The problem is reduced to an equivalent optimal control problem with the stationary value function. It is shown that the value function is the generalized minimax solution of the corresponding Hamilton–Jacobi equation. The boundary condition for the stationary value function is replaced by the property of the Hölder continuity and the sublinear growth condition. A backward procedure on infinite time horizon is proposed for construction of the value function. This procedure approximates the value function as the generalized minimax solution of the stationary Hamilton–Jacobi equation. Its convergence is based on the contraction mapping method defined on the family of uniformly bounded and Hölder continuous functions. After the special change of variables the procedure is realized in numerical finite difference schemes on strongly invariant compact sets for optimal control problems and differential games.

Keywords: Optimal control, Hamilton–Jacobi equations, numerical optimization methods, approximation schemes.

1. INTRODUCTION

In the analysis of optimal control problems on infinite horizon one should deal with the stationary Hamilton–Jacobi equations of the special type. Such kind of problems arise in economic growth models and in stabilization problems for dynamic systems. In this research, we deal with the case when a quality functional contains an unbounded integrand which is discounted by an exponential index. The boundary condition for the stationary value function is substituted by the property of the Hölder continuity and the sublinear growth condition. We study the problem of numerical construction of the value function for the optimal control problem as a generalized solution of the Hamilton-Jacobi equation. For this purpose we use the backward procedure of the dynamic programming principle and demonstrate that it can be interpreted as a contraction mapping method for the stationary Hamilton-Jacobi equation. Convergence of this method is proved for the family of uniformly bounded and Hölder continuous functions.

The backward procedures were originated in the dynamic programming principle in Bellman (1957) and were adjusted to construction of stable bridges in differential games in Krasovskii and Subbotin (1974). The practical implementation of backward procedures for construction of stable bridges was realized in Tarasyev, Ushakov and Khripunov (1987) basing on the concept of the stable absorption operator.

In this paper, based on backward procedures we develop the numerical methods for construction of value functions in optimal control problems with an unbounded integrand on infinite horizon by generalizing the approach proposed in Dolcetta (1983) for problems with a bounded integral functional. Let us note that this scheme was expanded later on for solution of differential games with bounded functionals on infinite horizon in Adiatulina and Tarasyev (1987). The existence result for the value function for problems with unbounded integrands was obtained in Nikolskii (2002). In parallel, optimal control problems with infinite time horizon are analyzed within modifications of transversality conditions for the Pontryagin maximum principle (see Aseev and Veliov (2015)).

In our analysis, we are based on the concept of generalized (nonsmooth) solutions of Hamilton-Jacobi equations (see Crandall and Lions (1983), Subbotin (1991)). Particularly, we use elements approximation schemes proposed in Souganidis (1985) and constructions of conjugate derivatives Subbotin and Tarasyev (1985). Also we apply results on properties and infinitesimal stability constructions for value functions in optimal control problems with infinite horizon Bagno and Tarasyev (2019).

10.1016/j.ifacol.2020.12.104

The special change of variables provides the possibility to realize the backward procedure on strongly invariant compact sets Aubin (1991) and, thus, to reduce calculations of approximation schemes similarly to tree-structure algorithms Alla, Falcone and Saluzzi (2019).

The paper has the next structure. Section 2 gives notations, definitions and statements that are necessary for describe results. Section 3 presents the conversion to equivalent optimal control problem with stationary value function and provides the description of the backward procedure. Section 4 introduces the change of variables that allows to reduce calculations to a compact strongly invariant set.

2. PROBLEM STATEMENT AND BASIC DEFINITIONS

We consider the optimal control problem with the dynamics

$$\dot{x}(t) = f(x(t), u(t)), \quad t \in [t_0, +\infty), \quad x(t_0) = x_0. \tag{1}$$

Here $t \in [t_0, +\infty]$, $x \in \mathbf{R}^n$ is phase vector, $u \in P \subset \mathbf{R}^p$ is control parameter, P is a compact set.

The quality functional is determined as

$$J(x(\cdot), u(\cdot)) = \int_{t_0}^{+\infty} e^{-\lambda\tau} g(x(\tau), u(\tau))\, d\tau, \tag{2}$$
$$\lambda > 0, \quad t_0 > 0.$$

We suppose that the following conditions are valid for the problem (1)–(2).

(1) Functions f and g are continuous on $\mathbf{R}^n \times P$.
(2) The Lipschitz condition in argument x takes place for all $x_1, x_2 \in \mathbf{R}^n$ and for all $u \in P$:
$$\|f(x_1, u) - f(x_2, u)\| \le L\|x_1 - x_2\|, \tag{3}$$
$$|g(x_1, u) - g(x_2, u)| \le L\|x_1 - x_2\|,$$
where L is a Lipschitz constant.
(3) The sublinear growth condition in argument x is true for all $x \in \mathbf{R}^n$, $u \in P$:
$$\|f(x, u)\| \le \kappa(1 + \|x\|), \tag{4}$$
$$|g(x, u)| \le \kappa(1 + \|x\|), \tag{5}$$
where $\kappa > 0$ is a constant.

The problem is to maximize the functional (2) on trajectories of the system (1) generated by measurable controls from the set U with values in the compact set U.

The value function in problem with infinity horizon is the function that matches each initial position (t_0, z_0), $t_0 \in (0, T)$, $z_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ $x_0 \in \mathbf{R}^n$, $y_0 \in R$, $x_0 = x(t_0)$, the largest value of the quality functional

$$\omega(t_0, z_0) = \limsup_{T \to +\infty} \left(y_0 + \int_{t_0}^{T} e^{-\lambda\tau} g(x(\tau), u(\tau))\, d\tau \right).$$

The value function has the following important properties under the condition $\lambda > \kappa$. It is continuous in the Hölder sense, and it satisfies the sublinear growth condition.

Claim 1. If $\lambda > \kappa$ then the next estimate for the value function is valid

$$|\omega(t, z)| \le A + B\|x\|,$$
$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad t \ge t_0, \quad x \in \mathbf{R}^n, \quad y \in \mathbf{R}, \tag{6}$$

where

$$A = |y| + \frac{\kappa}{\lambda} e^{-\lambda t}, \quad B = \frac{1}{\lambda - \kappa} e^{-\lambda t}.$$

Claim 2. The Hölder continuous condition is true: for all x_1 and x_2

$$|\omega(0, z_1) - \omega(0, z_2)| \le C\|x_1 - x_2\|^{\gamma} + |y_1 - y_2|, \tag{7}$$

where $C > 0$, $\gamma > 0$, $z_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$, $x_i \in \mathbf{R}^n$, $y_i \in R$, $i = 1, 2$.

Let us introduce the Hamiltonian function for the optimal control problem

$$H(x, s) = \frac{1}{\lambda} \min_{u \in P} (\langle s, f(x, u) \rangle + g(x, u)). \tag{8}$$

Here $x \in R^n$, $s \in R^n$.

For the value function $\varphi\colon \mathbf{R}^n \to \mathbf{R}$ we consider the Hamilton–Jacobi equation

$$-\varphi(x) + H(x, \nabla\varphi(x)) = 0, \quad x \in \mathbf{R}^n. \tag{9}$$

Here $\nabla\varphi(x)$ is vector of partial derivatives of function $\varphi(x)$. Let us note that commonly the Hamilton–Jacobi equation may not have smooth solutions. In order to give the definition of the generalized minimax solution (which is nonsmooth, generally speaking) Subbotin (1991) of equation (9), we introduce the definition of Dini derivatives

The lower (upper) Dini derivative of a continuous function $\omega(x)$ is determined by the relation

$$\partial_-\omega(x)|(d) = \inf_{\varepsilon(\cdot) \in \Delta} \underline{\lim}_{\delta \to 0} \frac{\omega(x + \delta d + \varepsilon(\delta)) - \omega(x)}{\delta}$$
$$\left(\partial_-\omega(x)|(d) = \sup_{\varepsilon(\cdot) \in \Delta} \overline{\lim_{\delta \to 0}} \frac{\omega(x + \delta d + \varepsilon(\delta)) - \omega(x)}{\delta}\right),$$

where $x \in \mathbf{R}^n$, $d \in \mathbf{R}^n$, Δ — functions class $\varepsilon(\cdot) : [0, +\infty) \to \mathbf{R}^n$, and $\lim\limits_{\delta \to 0} \frac{\|\varepsilon(\delta)\|}{\delta} = 0$.

Let us consider the auxiliary Hamiltonian

$$\overline{H}(t, x, s, m) = \begin{cases} e^{-t}|m| H(x, \dfrac{s}{e^{-t}|m|}), & m \ne 0, \\ \lim\limits_{m \to 0} e^{-t} m H(x, \dfrac{s}{e^{-t} m}), & m = 0, \end{cases}$$

where $t \ge 0$, $x \in \mathbf{R}^n$, $s \in \mathbf{R}^n$, $m \in \mathbf{R}$.

Denote by the symbol S the ball of the unit radius

$$\overline{S} = \{\overline{s} = (s_1, s_2) \in \mathbf{R}^n \times \mathbf{R}\colon \|\overline{s}\| = 1\}.$$

Let us introduce sets that determine dynamic possibilities of the system

$$A(x) = \{\overline{f} = (f_1, f_2) \in \mathbf{R}^n \times \mathbf{R}\colon \|f\| \le \sqrt{2}\kappa(1 + \|x\|)\},$$

Consider the Hamiltonian for the zero time $t = 0$

$$A_{up}(x, q_1, q_2) =$$
$$= \{\overline{f} \in A(x)\colon \langle f_1, q_1 \rangle + \langle f_2, q_2 \rangle \ge \overline{H}(0, x, q_1, q_2)\}$$
$$A_{down}(x, p_1, p_2) =$$
$$= \{\overline{f} \in A(x)\colon \langle f_1, p_1 \rangle + \langle f_2, p_2 \rangle \le \overline{H}(0, x, p_1, p_2)\},$$

where $q_1, p_1 \in \mathbf{R}^n$, $q_2, p_2 \in \mathbf{R}$.

Now we can provide the definition of the generalized minimax solution in terms of directional derivatives Subbotin (1991).

The generalized minimax solution of equation Hamilton–Jacobi (9) is the function $\varphi : \mathbf{R}^n \to \mathbf{R}$ which meets the Hölder continuity continuity (6), the sublinear growth condition (7) and satisfies differential inequalities

$$\min_{(d_1,d_2)\in A_{up}(x,q_1,q_2)} \{d_2 + \partial_-\varphi(x)|(d_1)\} - \varphi(x) \leq 0,$$
$$\forall x \in \mathbf{R}^n, \overline{q} = (q_1, q_2) \in \overline{S},$$
$$\max_{(d_1,d_2)\in A_{down}(x,p_1,p_2)} \{d_2 + \partial_+\varphi(x)|(d_1)\} - \varphi(x) \geq 0,$$
$$\forall x \in \mathbf{R}^n, \overline{p} = (p_1, p_2) \in \overline{S},$$

or differential inequalities in terms of conjugate derivatives Subbotin and Tarasyev (1985)

$$\sup_{d\in\mathbf{R}^n} \{\langle s, d\rangle - \partial_-\varphi(x)|(d)\} \geq -\varphi(x) + H(x, s),$$
$$\forall s \in \mathbf{R}^n, x \in \mathbf{R}^n,$$
$$\inf_{d\in\mathbf{R}^n} \{\langle s, d\rangle - \partial_+\varphi(x)|(d)\} \leq -\varphi(x) + H(x, s),$$
$$\forall s \in \mathbf{R}^n, x \in \mathbf{R}^n.$$

Let us remind that the notion of the generalized minimax solution is equivalent to the viscosity solution Crandall and Lions (1983). One can prove that function $\varphi(x)$ is the value function of the optimal control problem (1)–(2) if and only if it is the unique minimax solution of equation (9).

3. BACKWARD PROCEDURE

In order to introduce the backward procedure in the class of bounded functions we make the change of variables

$$\psi(x) = \frac{\varphi(x)}{M + N\|x\|}, \tag{10}$$

where $M > 0$ and $N > 0$ is some constants. Let us note that function $\dfrac{\varphi(x)}{M + N\|x\|}$ is not smooth at point $x = 0$. But it can be smoothed out in ε-neighborhood of zero, $\varepsilon > 0$. To do this, we substitute the nonsmooth function $\|x\|$ with its smooth approximation

$$r_\varepsilon(x) = \begin{cases} \|x\|, & \text{if} \|x\| > \varepsilon, \\ \|x\|^2/(2\varepsilon) + \varepsilon/2, & \text{if} \|x\| \leq \varepsilon. \end{cases} \tag{11}$$

We set $M = N = \kappa$ and substitute the change of variables in terms of function $\psi(x)$ to the Hamilton–Jacobi equation (9)

$$-\psi(x)(\kappa(1 + r_\varepsilon(x)))+$$
$$+\frac{1}{\lambda}\min_{u\in P}\Bigg(\langle \nabla\psi(x)(\kappa(1 + r_\varepsilon(x)))+$$
$$+\psi(x)\nabla r_\varepsilon(x), f(x, u)\rangle + g(x, u)\Bigg) = 0.$$

It can be rewritten in the form

$$\min_{u\in P}\Bigg(- \lambda\psi(x) + \langle\nabla\psi(x), f(x, u)\rangle+ \tag{12}$$
$$+\kappa\psi(x)\left\langle \nabla r_\varepsilon(x), \frac{f(x, u)}{\kappa(1 + r_\varepsilon(x))}\right\rangle + \frac{g(x, u)}{\kappa(1 + r_\varepsilon(x))}\Bigg) = 0,$$

that contains the Hamiltonian

$$\widehat{H}(x, \psi(x), \nabla\psi(x)) = \min_{u\in P}\Bigg(\langle\nabla\psi(x), f(x, u)\rangle +$$
$$+\kappa\psi(x)\left\langle \nabla r_\varepsilon(x), \frac{f(x, u)}{\kappa(1 + r_\varepsilon(x))}\right\rangle + \frac{g(x, u)}{\kappa(1 + r_\varepsilon(x))}\Bigg).$$

The equation (12) can be presented as

$$-\lambda\psi(x) + \widehat{H}(x, \psi(x), \nabla\psi(x)) = 0. \tag{13}$$

Function $\psi(x)$ can be approximated by function $\psi^h(x)$ in the Taylor expansion

$$\psi^h(x + hf(x, u)) - \psi^h(x) \approx h\langle\nabla\psi(x), f(x, u)\rangle.$$

Substituting this expression into equation (12) and multiplying it by h we obtain

$$\min_{u\in P}\Bigg(- \lambda h\psi^h(x + hf(x, u)) + \psi^h(x + hf(x, u))-$$
$$-\psi^h(x) + \kappa h\psi^h(x + hf(x, u))\left\langle \nabla r_\varepsilon(x), \frac{f(x, u)}{\kappa(1 + r_\varepsilon(x))}\right\rangle +$$
$$+\frac{hg(x, u)}{\kappa(1 + r_\varepsilon(x))}\Bigg) = 0,$$

or equivalently

$$\min_{u\in P}\Bigg(- \psi^h(x)+$$
$$+\left(1 - \lambda h + \kappa h\left\langle \nabla r_\varepsilon(x), \frac{f(x, u)}{\kappa(1 + r_\varepsilon(x))}\right\rangle\right)\cdot \tag{14}$$
$$\cdot\psi^h(x + hf(x, u))\frac{hg(x, u)}{\kappa(1 + r_\varepsilon(x))}\Bigg) = 0,$$

here $x \in \mathbf{R}^n$, $h > 0$.

Let us introduce notations

$$p(x, u, h, \lambda, \kappa) = 1 - \lambda h + \kappa h\left\langle \nabla r_\varepsilon(x), \frac{f(x, u)}{\kappa(1 + r_\varepsilon(x))}\right\rangle,$$

$$q(x, u, h, \kappa) = \frac{hg(x, u)}{\kappa(1 + r_\varepsilon(x))}$$

and substitute it in expression (14)

$$\min_{u\in P}(-\psi^h(x)+$$
$$+p(x, u, h, \lambda, \kappa)\psi^h(x + hf(x, u)) + q(x, u, h, \kappa)) = 0.$$

Let us introduce the operator

$$Q\psi(x) =$$
$$= \min_{u\in P}\Bigg(p(x, u, h, \lambda, \kappa)\psi(x + hf(x, u)) + q(x, u, h, \kappa)\Bigg).$$

Consider the time interval $[t_0, t_n]$, where t_0 is initial moment, t_n is some sufficiently large moment, and determine its partition $\{\Delta(t_i)\}$. Let us describe the method of calculation of values of the function ψ_Δ that approximates the value function ψ in the following backward procedure

$$\psi_\Delta(x_{n-1}) = Q\psi_\Delta(x_n), \quad \psi_\Delta(x_n) = \psi(x(t_n)). \tag{15}$$

(Souganidis, 1985, theorem 2.1) proved that such backward procedure approximates the generalized viscosity solution of equation (13) with accuracy

$$\|\psi_\Delta - \psi\| \le Ch^{1/2};$$

where C is a positive constant. In the mentioned paper, more general case of the Hamiltonian is considered and it contains the Hamiltonian of equation (13). Let us give the formulation of this result.

Claim 3. Denote by the symbol $C(X)$ the space of bounded real valued Lipschitz continuous functions defined on set X. Let $\psi \in C(\mathbf{R}^n \times [0,T])$ is the unique viscosity solution of (13) on $C(\mathbf{R}^n \times [0,T])$ for boundary condition $\psi_0 = \psi(x,0) \in C(\mathbf{R}^n)$ and function $H\colon [0,T] \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$, satisfies the following conditions:

(1) the Hamiltonian H is uniformly continuous on $[0,T] \times \mathbf{R}^n \times [-R,R] \times B_n(0,R)$, where $B_n(x_0,R) = \{x \in \mathbf{R}^n \colon |x - x_0| \le R\}$,

(2) there is a constant $C > 0$ such that
$$C = \sup_{(x,t)\in\overline{Q}_T} |H(t,x,0,0)| < \infty,$$
where $\overline{Q}_T = \mathbf{R}^n \times [0,T]$,

(3) for $R \ge 0$ there is a $C_R > 0$ such that
$$|H(t,x,r,p) - H(t,y,r,p)| \le C_R(1 + |p|)|x - y|$$
for $t \in [0,T]$, $|r| \le R$, x, y, $p \in \mathbf{R}^n$,

(4) for $R > 0$ there is a constant $\overline{L}_R > 0$, depending on R such that
$$|H(t,x,r,p) - H(t,x,s,p)| \le \overline{L}_R(r - s)$$
for $x \in \mathbf{R}^n$, $-R \le s \le r \le R$, $0 \le t \le T$, $p \in \mathbf{R}^n$,

(5) for $R \ge 0$ there is a $N_R > 0$ such that
$$|H(t,x,r,p) - H(\bar{t},x,r,p)| \le N_R(1 + |p|)|t - \bar{t}|$$
for t, $\bar{t} \in [0,T]$, $|r| \le R$, x, $p \in \mathbf{R}^n$,

(6) for $R \ge 0$ there is a $M_R > 0$ such that
$$|H(t,x,r,p) - H(t,x,r,q)| \le M_R|p - q|$$
for $t \in [0,T]$, $x \in \mathbf{R}^n$, $|r|$, $|p|$, $|q| \le R$, p, $q \in \mathbf{R}^n$.

For the pair $(t,\rho) \in \{[0,T] \times [0,\rho_0] \colon 0 \le \rho \le t\}$, where $\rho_0 = \rho_0(\|\psi_0\|) > 0$, let function $F(t,\rho,\cdot,\cdot)\colon C(\mathbf{R}^n) \times C(\mathbf{R}^n) \to C(\mathbf{R}^n)$ be such that for every $\psi, \overline{\psi}, \xi, \overline{\xi} \in C(\mathbf{R}^n)$

(1) $F(t,0,\psi,\xi) = \xi$,

(2) the mapping $(t,\rho) \to F(t,\rho,\psi,\psi)$ is continuous,

(3) $F(t,\rho,\psi,\xi + k) = F(t,\rho,\psi,\xi) + k$ for every $k \in \mathbf{R}$,

(4) $\|F(t,\rho,\psi,\psi) - \psi\| \le C_1$, where a constant $C_1 = C_1(\|\psi\|, \|D\psi\|) \ge 0$,

(5) there exists an $r > 0$ and $L_1 > 0$ such that if $\xi(x) \le \overline{\xi}(x)$ for every $x \in \mathbf{R}^n$, then for any $y \in \mathbf{R}^n$ such that
$$|\xi(y+w) - \xi(y+\overline{w})|,\ |\overline{\xi}(y+w) - \overline{\xi}(y+\overline{w})| \le \overline{L}|w - \overline{w}|$$
for every w, $\overline{w} \in \{x \in \mathbf{R}^n \colon \|x\| \le \rho r\}$
$$F(t,\rho,\psi,\xi)(y) \le F(t,\rho,\psi,\overline{\xi})(y),$$
where $L = \sup_{0\le\tau\le T} \|D\psi(\cdot,\tau)\|$ and $\overline{L} = \max(L_1, L) + 1$,

(6) there exists a constant C_2 such that
$$\|F(t,\rho,\psi,\psi)\| \le e^{\rho C_2}(\|\psi\| + \rho C_2),$$
provided that $\|D\psi\| \le \overline{L}$,

(7) there exist constants C_3, $C_4 > 0$ such that
$$e^{T(C_3+C_4)}(\|D\psi_0\| + TC_4) \le \overline{L}$$
and
$$\|DF(t,\rho,\psi,\psi)\| \le e^{\rho(C_3+C_4)}(\|D\psi\| + \rho C_4),$$
provided that $\|\psi\| \le e^{TC_2}(\|\psi_0\| + TC_2)$ and $\|D\psi\| \le \overline{L}$,

(8) for each φ in the space of twice differentiable on $\mathbf{R}^n$ functions with bounded partial derivatives of the first and second order such that $|D\varphi(x)| < L + 1$, and for every $x \in \mathbf{R}^n$ the following relations take place
$$\left|\frac{F(t,\rho,\psi,\varphi)(x) - \varphi(x)}{\rho} + H(t,x,\psi(x),D\varphi(x))\right| \le$$
$$\le C_5(1 + \|D\varphi\| + \|D^2\varphi\|)\rho,$$
where a constant $L = \sup_{0\le\tau\le T} \|D\psi(\cdot,\tau)\|$ and a constant $C_5 = C_5(\|\psi\|, \|D\psi\|, \overline{L})$.

For a partition $\Delta = \{0 = t_0 < t_1 < \cdots < t_{n(\Delta)} = T\}$ of $[0,T]$ let $\psi_\Delta \colon \mathbf{R}^n \times [0,T] \to \mathbf{R}$ be defined by

$$\psi_\Delta(x,0) = \psi_0(x),$$
$$\psi_\Delta(x,t) = F(t, t - t_{i-1}, \psi_\Delta(\cdot,t_{i-1}), \psi_\Delta(\cdot,t_{i-1}))(x)$$
$$t \in (t_{i-1}, t_i],\ i = 1,\ldots,n(\Delta).$$

Then there exists a constant $C > 0$, depending only on $\|\psi_0\|$ and $\|D\psi_0\|$ such that

$$\|\psi_\Delta - \psi\| \le C|h|^{1/2}$$

for $|h|$ sufficiently small.

Basing on this result one can obtain the estimate of convergence for the backward procedure for approximation of the value function in optimal control problem with infinite horizon. Namely, one can prove the following theorem.

Theorem 4. Let $\lambda > \kappa$, $C(T) = \overline{A}e^{\kappa T}$, where $\overline{A}$ is a positive constant, $\overline{A} > 0$. Then the backward procedure (15) approximates function ψ on interval $[t_0, +\infty)$ with the estimate

$$|h|^{\frac{\lambda}{2(\kappa+\lambda)}} \left(\frac{1}{\kappa} + \frac{1}{\lambda}\right) \left(\overline{A}\kappa\right)^{\frac{\lambda}{\kappa+\lambda}}.$$

Proof. We have

$$\int_T^{+\infty} \psi(x(\tau))\, d\tau = \int_T^{+\infty} \frac{e^{-\lambda\tau} g(x(\tau),u(\tau))}{\kappa(1 + \|x(\tau)\|)}\, d\tau$$

by definition of the value function 2 and according to substitution (10). Since function $g(x,u)$ satisfies inequality (5) then

$$\int_T^{+\infty} \|\psi(x(\tau))\|\, d\tau \le \int_T^{+\infty} e^{-\lambda\tau}\, d\tau = \frac{1}{\lambda} e^{-\lambda T}. \tag{16}$$

The backward procedure approximates function ψ on interval $[t_0, T]$ with the accuracy

$$\|\psi_\Delta - \psi\| \le C(T)|h|^{1/2}$$

according to statement 3.

Let us add to this expression the estimate for the "tail" (16) of function ψ. We show that the next relation is true on the half axis $[t_0, +\infty)$

$$\|\psi_\Delta - \psi\| \leq C(T)|h|^{1/2} + \frac{1}{\lambda}e^{-\lambda T} = \overline{A}e^{\kappa T}|h|^{1/2} + \frac{1}{\lambda}e^{-\lambda T}. \tag{17}$$

The relation depends on the time moment which is the terminal time of the backward procedure. We can improve the relation if we find the time moment where the relation reaches the lowest value. For this, let us calculate the derivative of the sum on the right hand side (17) and equate it to zero

$$(C(T)|h|^{1/2} + \frac{1}{\lambda}e^{-\lambda T})' = (\overline{A}e^{\kappa T}|h|^{1/2} + \frac{1}{\lambda}e^{-\lambda T})' =$$
$$= \overline{A}\kappa e^{\kappa T}|h|^{1/2} - e^{-\lambda T} = 0.$$

The solution of this equation can be presented as follows

$$T_* = \frac{-\ln(\overline{A}\kappa|h|^{1/2})}{\kappa + \lambda}.$$

This value delivers the minimum value for the right hand side (17) according to sufficient conditions of minimum. Let us check this. Really, the second derivative of the right hand side equals

$$(C|h|^{1/2} + \frac{1}{\lambda}e^{-\lambda T})'' = (\overline{A}e^{\kappa T}|h|^{1/2} + \frac{1}{\lambda}e^{-\lambda T})'' =$$
$$= \overline{A}\kappa^2 e^{\kappa T}|h|^{1/2} + \lambda e^{-\lambda T}.$$

This expression is positive.

Let us substitute point T_* into relation (17), which is valid for all T and, specifically, it holds for the minimum point T_*

$$\|\psi_\Delta - \psi\| \leq \overline{A}|h|^{1/2}e^{\kappa T_*} + \frac{1}{\lambda}e^{-\lambda T_*} =$$
$$\leq \overline{A}|h|^{1/2}e^{\dfrac{-\kappa\ln(\overline{A}\kappa|h|^{1/2})}{\kappa+\lambda}} + \frac{e^{\dfrac{\lambda\ln(\overline{A}\kappa|h|^{1/2})}{\kappa+\lambda}}}{\lambda} =$$
$$= \frac{\overline{A}\kappa|h|^{1/2}\left(\overline{A}\kappa|h|^{1/2}\right)^{\frac{-\kappa}{\kappa+\lambda}}}{\kappa} + \frac{\left(\overline{A}\kappa|h|^{1/2}\right)^{\frac{\lambda}{\kappa+\lambda}}}{\lambda} =$$
$$= \frac{\left(\overline{A}\kappa|h|^{1/2}\right)^{\frac{\lambda}{\kappa+\lambda}}}{\kappa} + \frac{\left(\overline{A}\kappa|h|^{1/2}\right)^{\frac{\lambda}{\kappa+\lambda}}}{\lambda}.$$

Theorem is proved.

4. REDUCTION OF THE BACKWARD PROCEDURE TO A COMPACT STRONGLY INVARIANT SET

Let us note that the backward procedure is determined on the whole phase space, $x \in R^n$. To reduce calculations, we make the following change of variables

$$y(t) = e^{-\kappa(t-t_0)}x(t). \tag{18}$$

After change of variables we get the following system of differential equations

$$\dot{y}(t) = -\kappa y(t) + e^{-\kappa(t-t_0)}f(e^{\kappa(t-t_0)}y(t), u(t)),$$
$$J(y(\cdot), u(\cdot)) = \int_{t_0}^{+\infty} e^{-\lambda\tau}g(e^{\kappa(\tau-t_0)}y(\tau), u(\tau))\,d\tau,$$
$$t \in [t_0, +\infty), \quad y(t_0) = y_0 = x_0, \quad t_0 > 0, \quad \lambda > 0.$$

One can see that values of variable $y = y(t)$ are contained in a compact set due to conditions (3), (4) and according to results for strongly invariant sets Aubin (1991).

To convert the problem to the stationary one, let us introduce the auxiliary variable $\xi(t) = e^{-\kappa(t-t_0)}$, which allows to consider the stationary dynamic system

$$\dot{y} = -\kappa y + \xi f(\frac{y}{\xi}, u), \tag{19}$$
$$\dot{\xi} = -\kappa\xi,$$
$$J(y, u) = \int_{t_0}^{+\infty} e^{-\lambda\tau}g(\frac{y}{\xi}, u)\,d\tau,$$
$$y(t_0) = x_0, \quad \xi(t_0) = 1, \quad \lambda > 0.$$

Finally, let us note that the dynamic system (19) has the stationary form and its trajectories survive in compact convex sets (i.e. balls, rectangular parallelepipeds) with the center at the origin. Hence, the proposed backward procedure can be implemented in these compacts convex sets instead of the whole space $\mathbf{R^n}$, and this fact can significantly reduce calculations of approximation schemes.

5. CONCLUSIONS AND FUTURE WORK

The paper deals with approximation schemes for construction of value functions as generalized solutions of stationary Hamilton–Jacobi equations in optimal control problems with infinite time horizon. One of basic elements of the considered optimal control problem is a quality functional with an unbounded integrand which is discounted by an exponential index. Such statements arise in models of economic growth and problems of stabilization of dynamic systems.

The value function for such statement of the optimal control problem is the generalized minimax solution of the corresponding stationary Hamilton–Jacobi equation. The boundary condition for the stationary value function is replaced by the property of the Hölder continuity and the sublinear growth condition. Basing on these properties, we develop a backward approximation procedure on infinite time horizon for construction of the value function. The convergence of this backward approximation procedure is based on the contraction mapping method defined on the family of uniformly bounded and Hölder continuous functions.

In this paper, we provide the accuracy estimate for the proposed approximation procedure and calculate precisely parameters for this estimate, which ensures the convergence of approximations to the value function at a rate of the same order as the exponent of the Hölder condition.

We introduce also the special change of variables in the optimal control problem with infinite time horizon, which reduces the backward approximation procedure for construction of the value function to a compact strongly invariant set.

In the future research, it is planned to apply the proposed approach for construction of value functions in economic growth models with exhaustible resources.

REFERENCES

Adiatulina R.A., Tarasyev A.M. A differential game of unlimited duration, *J. Appl. Math. Mech.*, volume 51, no. 4, pages 415–420, 1987.

Alla A., Falcone M., Saluzzi L. An efficient DP algorithm on a tree-structure for finite horizon optimal control problems, *SIAM Journal on Scientific Computing*, volume 41, no. 4, pages A2384-A2406, 2019.

Aubin J.-P. *Viability Theory*, Springer Science+Business Media, LLC, 1991.

Aseev S.M., Veliov V.M. Maximum principle for infinite-horizon optimal control problems under weak regularity assumptions, *Proc. Steklov Inst. Math.*, volume 291, no. 1, pages 22–39, 2015.

Bagno A.L., Tarasyev A.M. Estimate for the Accuracy of a Backward Procedure for the HamiltonJacobi Equation in an Infinite-Horizon Optimal Control Problem, *Proc. Steklov Inst. Math.*, volume 304, pages 110–123, 2019.

Bellman R. *Dynamic programming*, New York: Princeton University Press, 1957.

Crandall M.G., Lions P.-L. Viscosity solutions of Hamilton–Jacobi equations, *Trans. Amer. Math. Soc.*, volume 277, no. 1, pages 1–42, 1983.

Dolcetta I.C. On a discrete approximation of the Hamilton–Jacobi equation of dynamic programming, *Appl. Math. Optimiz.*, volume 10, no. 4, pages 367–377, 1983.

Nikolskii M.S., Investigation of the continuity and Lipschitz properties for the Bellman function in some optimization problems on the semi-infinite interval $[0, +?)$, *Differential Equations*, volume 38, no. 11, pages 1599–1604, 2002.

Krasovskii N.N., Subbotin A.I. *Pozitsionnye differentsial'nye igry* (Positional differential games), Moscow: Nauka, 1974 (in Russian).

Souganidis P.E. Approximation schemes for viscosity solutions of Hamilton–Jacobi equations, *J. Differential Equations*, volume 59, no. 1, pages 1–43, 1985.

Subbotin A.I. *Minimaksnye neravenstva i uravneniya Gamil'tona–Yakobi* (Minimax inequalities and Hamilton–Jacobi equations), Moscow, Nauka Publ., 1991 (in Russian).

Subbotin A.I., Tarasyev A.M. Conjugate derivatives of the value function of a differential game, *Soviet Math. Dokl.*, volume 283, no. 3, pages 559–564, 1985 (in Russian).

Tarasyev A.M., Ushakov V.N., Khripunov A.P. On a numerical algorithm for solving game-theoretical problems of control, *J. Appl. Math. Mech.*, volume 51, no. 2, pages 216–222, 1987 (in Russian).